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ON THE CHOICE OF THE EXTERIOR KNOTS IN THE B-SPLINE BASIS FOR A--ETC(U)
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ON THE CHOICE OF THE EXTERIOR KNOTS IN THE B-SPLINE BASIS FOR A SPLINE SPACE

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ABSTRACT

The B-spline representation of a spline on some interval $[a,b]$ requires the introduction of additional knots outside $]a,b[$ that have nothing in common with the spline space itself. In this note, it is shown that locating all these additional knots at the endpoints a and b of the interval minimizes the matrix norm $\|\cdot\|_\infty$ of the matrix $(\lambda_i N_{j,k})^{-1}$ where λ_i are given linear interpolation conditions, and so is preferable when the B-spline representation of a spline interpolant is to be constructed. Such a choice usually also simplifies the algorithms. In particular, one is able to compute stably the B-spline coefficients of a complete spline interpolant by Gauss elimination without pivoting though the corresponding matrix fails to be totally positive or diagonally dominant.

AMS(MOS) Subject Classification - 65D10, 41A15

Key Words: B-spline basis, exterior knots, spline interpolation

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SIGNIFICANCE AND EXPLANATION

The B-spline representation of a spline on some interval $[a,b]$ requires the introduction of additional knots outside $]a,b[$ that have nothing in common with the spline space itself. Nevertheless, their choice influences the numerical accuracy when the B-spline representation

$$\sum \alpha_i N_{i,k,\underline{t}}$$

of a spline approximant is to be constructed. In this note we prove that locating all the additional knots at the endpoints a and b of the interval minimizes the norm

$$\|(\lambda_i N_{j,k,\underline{t}})^{-1}\|_\infty .$$

and is therefore preferable. Here $\underline{\lambda} := (\lambda_i)$ is an arbitrary sequence of linear functionals with support in I except for the assumption that the interpolation problem is correct, i.e. $(\lambda_i N_{j,k,\underline{t}})$ nonsingular. The above choice usually also simplifies the algorithms. In particular, one is able to compute in a stable way the B-spline coefficients of a complete spline interpolant by Gauss elimination without pivoting though the corresponding matrix fails to be totally positive or diagonally dominant.

The proof relies on the observation that

$$(N_{i,k,\underline{\tau}}) = (N_{i,k,\underline{t}})Q_{\underline{t}\,\underline{\tau}} \quad \text{on} \quad [a,b] ,$$

with $\underline{\tau},\underline{t}$ being two knot sequences that coincide in $[a,b]$. The matrix $Q_{\underline{t}\,\underline{\tau}}$ for the particular choice of additional knots $\underline{t}$ described above turns to be totally positive and its columns sum up to 1, i.e. $Q_{\underline{t}\,\underline{\tau}}$ is composed of (discrete) B-splines.

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The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

ON THE CHOICE OF THE EXTERIOR KNOTS IN THE B-SPLINE BASIS FOR A SPLINE SPACE

J. KOZAK

1. The Result.

In a practical computation one is rarely able to make statements about the inverse of a given matrix, particularly if the linear system to be solved depends on several free parameters. This note is intended to demonstrate that in an important linear problem, that of spline interpolation, the properties of a chosen basis allow us to draw an interesting conclusion.

To start, let $I := [a,b] \subset R$ be a given real interval, partitioned by the sequence

$$a =: t_k < t_{k+1} \leq \cdots \leq t_n < t_{n+1} := b\ , \qquad (1.1)$$

with $t_i < t_{i+k}$, all i and some integer $1 \leq k \leq n$. For the purpose of using B-splines, the sequence is extended by

$$t_1 \leq t_2 \leq \cdots \leq t_{k-1} \leq t_k\ ,\quad t_{n+1} \leq t_{n+2} \leq \cdots \leq t_{n+k}\ . \qquad (1.2)$$

To simplify the distinction between both parts of $\underline{t} := (t_i)_{i=1}^{n+k}$ put $\mathrm{int}(\underline{t}) := (t_i)_{i=k}^{n+1}$, $\mathrm{ext}(\underline{t}) := (t_i)_{i=1;i=n+2}^{k-1;n+k}$. The collection of polynomial splines of order k on I with knot sequence $\underline{t}$ is defined by

$$S_{k,\underline{t}}(I) := \{f \mid f_{|]t_i,t_{i+1}[} \text{ is a polynomial of degree} < k\ ,$$
$$\mathrm{jump}_{t_i} f^{(r)} = 0, r < k - \mathrm{card}\{j \mid t_j = t_i\},\ \text{all } i\}\ .$$

Any $f \in S_{k,\underline{t}}(I)$ admits a unique B-spline representation

$$f = \sum_{i=1}^{n} \alpha_i N_{i,k,\underline{t}} \qquad (1.3)$$

that has proved very successful in practical computations. Here, $(N_{i,k,\underline{t}})$ is the B-spline partition of the unity, i.e.

$$N_{i,k,\underline{t}}(x) := (t_{i+k} - t_i)[t_i,t_{i+1},\cdots, t_{i+k}](\cdot - x)_+^{k-1}\ .$$

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Consider now the following interpolation problem: for given interpolation conditions $\underline{\lambda} := (\lambda_i)_{i=1}^n$ and prescribed numbers $r := (r_i)_{i=1}^n \in R^n$ find $f \in S_{k,\underline{t}}(I)$ such that

$$\lambda_i f = r_i \ , \quad \text{all} \quad i \ . \tag{1.4}$$

There $\underline{\lambda}$ is an arbitrary sequence of linear functionals with support in I except for the assumption that the interpolation problem is correct, i.e.

$$f \in S_{k,\underline{t}}(I) \quad \text{and} \quad \lambda_i f = 0 \ , \quad \text{all} \quad i \ , \quad \text{implies} \quad f = 0 \ .$$

The particular representation (1.3) leads to the solution

$$\underline{\alpha} := (\alpha_i)_{i=1}^n = A_{\underline{t}}^{-1} \, \underline{r} \tag{1.5}$$

with $A_{\underline{t}} := (\lambda_i N_{j,k,\underline{t}})_{i,j=1}^n$. We note that the solution f does not depend on $\text{ext}(\underline{t})$, but $A_{\underline{t}}$ and consequently $\underline{\alpha}$ do. This fact can influence the numerical accuracy of the computed spline.

Cox [8] considered various ways of choosing the additional knots. As he concluded from numerical evidence, in the case of general-purpose algorithms the coincident choice at end points is preferable. In fact, he observed in [7] that the spectral condition number of the matrix obtained in particular spline least-squares problems is considerably smaller for

$$\text{ext}(\underline{t}) = (\underbrace{t_k, t_k, \cdots, t_k}_{k-1}, \ \underbrace{t_{n+1}, t_{n+1}, \cdots, t_{n+1}}_{k-1}) \tag{1.6}$$

compared with an equidistant and an average choice. We note also that in the book by de Boor [5] the choice (1.6) is the rule.

Theorem 1.1. Let the knot sequences $\underline{\tau}, \underline{t}$ satisfy

$$\text{int}(\underline{\tau}) = \text{int}(\underline{t}),$$

$$\tau_i < t_i \quad , \quad i = 1,2,\cdots,k-1,$$

$$\tau_i > t_i \quad , \quad i = n+2,n+3,\cdots,n+k \ ,$$

and let $n > 2(k-1)$. If the interpolation problem (1.4) is correct, then

$$\|A_{\underline{\tau}}^{-1}\|_\infty > \|A_{\underline{t}}^{-1}\|_\infty \ . \tag{1.7}$$

The inequality (1.7) displays the numerical advantage of using the ext($\underline{t}$) as defined in (1.6). Also, following [2], consider the interpolation map

$$P: R^n \to S_{k,\underline{t}}(I) : \underline{r} \to P\underline{r} := (\lambda_i P\underline{r} = r_i \ , \ \text{all} \ i \ ).$$

Then

$$\|P\| := \sup_{\|\underline{r}\|_\infty \le 1} (\sup_{x\in I}|P\underline{r}(x)|) = \sup_{\|\underline{r}\|_\infty \le 1} (\sup_{x\in I}|\sum_{i=1}^{n} \alpha_i N_{i,k,\underline{t}}(x)|) \le \|A_{\underline{t}}^{-1}\|_\infty \tag{1.8}$$

and (1.6) gives the best bound of the form (1.8). It is also clear from [2] that $A_{\underline{\tau}}$ must become singular, as $\tau_1 \to -\infty$ or $\tau_{n+k} \to \infty$. We shall explicitly observe

$$\|A_{\underline{\tau}}^{-1}\|_\infty \ge \frac{1}{\|A_{\underline{t}}\|_\infty} \max\left(\prod_{r=2}^{k-1} \frac{t_{k+1}-\tau_r}{t_{k+1}-\tau} \ , \ \prod_{r=2}^{k-1} \frac{\tau_{n+r}-t_n}{t_{n+r}-t_n} \right) . \tag{1.9}$$

Our final observation concerns complete even order spline interpolation. Let $k = 2m$ and

$$\underline{\lambda} := (\delta_{t_k}, \delta_{t_k}^{(1)}, \cdots, \delta_{t_k}^{(m-1)}, \delta_{t_{k+1}}, \cdots, \delta_{t_n}, \delta_{t_{n+1}}^{(m-1)}, \cdots, \delta_{t_{n+1}}) \tag{1.10}$$

with $\delta_t^{(r)} f := f^{(r)}(t)$. It is proved in [4] that *the Gauss elimination without pivoting can* be applied safely if the matrix is totally positive. Unfortunately for $\underline{\lambda}$ given in (1.10) the matrix is not totally positive. But (1.6) gives us $A_{\underline{t}}$ of the form

$$
\begin{matrix} m\left\{\begin{matrix}\\ \\ \end{matrix}\right. \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{matrix}
\begin{pmatrix}
x & & & & & & & & & & \\
x & x & & & & & & & & & \\
x & x & x & & & & & & 0 & & \\
x & x & x & x & x & & & & & & \\
 & x & x & x & x & x & & & & & \\
 & & & \ddots & & & & & & & \\
 & & & & & & & & & & \\
 & & & & & & & & & & \\
 & & & & & x & x & x & x & x & \\
 & & & & & & x & x & x & x & x \\
 & & & 0 & & & & & x & x & x \\
 & & & & & & & & & x & x \\
 & & & & & & & & & & x
\end{pmatrix}
\begin{matrix} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \left.\begin{matrix}\\ \\ \end{matrix}\right\}m \end{matrix}
$$

and we need to factor only the submatrix

$$
\underline{A}_t \begin{pmatrix} m+1, m+2, \cdots, n-m \\ m+1, m+2, \cdots, n-m \end{pmatrix},
$$

and this submatrix <u>is</u> totally positive.

2. Its Proof.

Consider two knot sequences $\underline{t}$, $\underline{\tau}$ that satisfy (1.1), (1.2). Assume that int $(\cdot)$ is strictly increasing. The general case will follow from continuity properties of divided differences. Of course there exists a matrix $Q_{\underline{t}\,\underline{\tau}}$ such that

$$A_{\underline{\tau}} = A_{\underline{t}}\, Q_{\underline{t}\,\underline{\tau}}\,, \tag{2.1}$$

but it is not so obvious that we can find $Q_{\underline{t}\,\underline{\tau}}$ explicitly when $\text{int}(\underline{\tau}) = \text{int}(\underline{t})$. Recall Marsden's identity [9]

$$(y-x)^{k-1} = \sum_{i=1}^{n} \varphi_{i,k,\underline{t}}(y)\, N_{i,k,\underline{t}}(x)\,, \qquad x,y \in I. \tag{2.2}$$

Here

$$\varphi_{i,k,\underline{t}} := \prod_{\ell=1}^{k-1} (\cdot - t_{\ell+i})\,. \tag{2.3}$$

As observed in [1], (2.2) implies

$$(y-x)_+^{k-1} = \sum_{i=1}^{n} \prod_{\ell=1}^{k-1} (y - t_{\ell+i})_+\, N_{i,k,\underline{t}}(x)\,, \quad y \in \text{int}(\underline{t})\,,\ x \in I. \tag{2.4}$$

But (2.4) holds for an $y < t_k = a$ too, since then both sides vanish identically. Hence

$$[y_0 y_1,\cdots,y_k](\cdot - x)_+^{k-1} = \sum_{i=1}^{n} [y_0,y_1,\cdots,y_k] \prod_{\ell=1}^{k-1} (\cdot - t_{\ell+i})_+ N_{i,k,\underline{t}}(x) \quad \text{on } I \tag{2.5}$$

under the assumption

$$y := (y_i)_{i=0}^{k} \subset \text{int}(\underline{t}) \cup\ ]-\infty,t_k[\,.$$

Similarly, with $\underline{y} \subset \text{int}(\underline{t}) \cup\ ]t_{n+1},\infty[$,

$$[y_0,y_1,\cdots,y_k](\cdot - x)_+^{k-1} = [y_0,y_1,\cdots,y_k](x - \cdot)_+^{k-1} =$$

$$= \sum_{i=1}^{n} [y_0,y_1,\cdots,y_k] \prod_{\ell=1}^{k-1} (t_{\ell+i} - \cdot)_+\, N_{i,k,\underline{t}}(x) \quad \text{on } I. \tag{2.6}$$

One finds that (2.5), (2.6) are the identity [3, (5.10)], adjusted to the finite interval I.

<u>Lemma 2.1.</u> Let $n > 2(k-1)$, $\text{int}(\underline{\tau}) = \text{int}(\underline{t})$. Then

$$(N_{i,k,\underline{\tau}})_{i=1}^{n} = (N_{i,k,\underline{t}})_{i=1}^{n}\, Q_{\underline{t}\,\underline{\tau}} \tag{2.7}$$

with

$$Q_{\underline{t}\,\underline{\tau}} := (q_{ij,\underline{t}\,\underline{\tau}})_{i,j=1}^{n}$$

and

$$q_{ij,\underline{t}\,\underline{\tau}} := \begin{cases} (\tau_{j+k} - \tau_j) \sum_{\ell=i}^{j} \varphi_{i,k,\underline{t}}(t_{\ell+k})/\varphi'_{j-1,k+2,\underline{\tau}}(\tau_{\ell+k}), & 1 \le i \le j \le k-1, \\ (\tau_{j+k} - \tau_j) \sum_{\ell=j}^{i} \varphi_{i,k,\underline{t}}(t_\ell)/\varphi'_{j-1,k+2,\underline{\tau}}(\tau_\ell), & n-k+2 \le j \le i \le n, \\ \delta_{i-j}, & \text{otherwise.} \end{cases}$$

Proof. Assume for a moment that $\text{int}(\underline{t})$ is strictly increasing. Choose $1 \le j \le k-1$. Since $n+1 \ge 2(k-1) + 1 = 2k-1$ we can use (2.5) for any such j to obtain

$$N_{j,k,\underline{\tau}} = \sum_{i=1}^{n} (\tau_{j+k} - \tau_j)[\tau_j,\tau_{j+1},\cdots,\tau_{j+k}] \prod_{\ell=1}^{k-1} (\cdot - t_{\ell+i})_+ N_{i,k,\underline{t}} \quad \text{on } I ,$$

and further

$$(\tau_{j+k} - \tau_j)[\tau_j,\tau_{j+1},\cdots,\tau_{j+k}] \prod_{\ell=1}^{k-1} (\cdot - t_{\ell+i})_+ =$$

$$= (\tau_{j+k} - \tau_j) \sum_{r=j}^{j+k} \frac{\prod_{\ell=1}^{k-1} (\tau_r - t_{\ell+i})_+}{\prod_{\substack{\ell=j \\ \ell\ne r}}^{j+k} (\tau_r - \tau_\ell)}$$

$$= 0 + (\tau_{j+k} - \tau_j) \sum_{r=i+k}^{j+k} \frac{\prod_{\ell=1}^{k-1} (\tau_r - t_{\ell+i})}{\prod_{\substack{\ell=j \\ \ell\ne r}}^{j+k} (\tau_r - \tau_\ell)} = q_{ij,\underline{t}\,\underline{\tau}}, \quad \text{all } i .$$

The proof for $n-k + 2 \le j \le n$ follows in the same way from (2.6), and (2.7) obviously holds for the remaining range of j. The proof is completed for a strictly increasing $\text{int}(\underline{t})$, but $\text{int}(\underline{\tau}) = \text{int}(\underline{t})$ simplifies $q_{ij,\underline{t}\,\underline{\tau}}$, say for the range $1 \le i \le j \le k-1$, to

$$q_{ij,\underline{t}\,\underline{\tau}} = (\tau_{j+k} - \tau_j)[\tau_{i+k},\tau_{i+k+1},\cdots,\tau_{j+k}]\Big(\prod_{\ell=i+1}^{k-1} (\cdot - t_\ell)/\prod_{\ell=j}^{k-1} (\cdot - \tau_\ell)\Big) , \quad (2.8)$$

and the general case follows. ∎

2. *Its Proof.*

Consider two knot sequences $\underline{t}$, $\underline{\tau}$ that satisfy (1.1), (1.2). Assume that int $(\cdot)$ is strictly increasing. The general case will follow from continuity properties of divided differences. Of course there exists a matrix $Q_{\underline{t}\,\underline{\tau}}$ such that

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but it is not so obvious that we can find $Q_{\underline{t}\,\underline{\tau}}$ explicitly when $\mathrm{int}(\underline{\tau}) = \mathrm{int}(\underline{t})$. Recall Marsden's identity [9]

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$$(y-x)_+^{k-1} = \sum_{i=1}^{n} \prod_{\ell=1}^{k-1} (y - t_{\ell+i})_+\, N_{i,k,\underline{t}}(x)\ , \quad y \in \mathrm{int}(\underline{t})\ ,\ x \in I\,. \tag{2.4}$$

But (2.4) holds for an $y \leq t_k = a$ too, since then both sides vanish identically. Hence

$$[y_0 y_1,\cdots,y_k](\cdot - x)_+^{k-1} = \sum_{i=1}^{n} [y_0,y_1,\cdots,y_k] \prod_{\ell=1}^{k-1} (\cdot - t_{\ell+i})_+ N_{i,k,\underline{t}}(x) \quad \text{on } I \tag{2.5}$$

under the assumption

$$\underline{y} := (y_i)_{i=0}^{k} \subset \mathrm{int}(\underline{t}) \cup\,]-\infty,t_k[\ .$$

Similarly, with $\underline{y} \subset \mathrm{int}(\underline{t}) \cup\,]t_{n+1},\infty[$,

$$[y_0,y_1,\cdots,y_k](\cdot - x)_+^{k-1} = [y_0,y_1,\cdots,y_k](x - \cdot)_+^{k-1} =$$

$$= \sum_{i=1}^{n} [y_0,y_1,\cdots,y_k] \prod_{\ell=1}^{k-1} (t_{\ell+i} - \cdot)_+\, N_{i,k,\underline{t}}(x) \quad \text{on } I. \tag{2.6}$$

One finds that (2.5), (2.6) are the identity [3, (5.10)], adjusted to the finite interval I.

<u>Lemma 2.1.</u> Let $n \geq 2(k-1)$, $\mathrm{int}(\underline{\tau}) = \mathrm{int}(\underline{t})$. Then

$$(N_{i,k,\underline{\tau}})_{i=1}^{n} = (N_{i,k,\underline{t}})_{i=1}^{n}\, Q_{\underline{t}\,\underline{\tau}} \tag{2.7}$$

with

In order to investigate $Q_{\underline{t}\,\underline{\tau}}$ further we compute, for $1 \le i \le j \le k-1$,

$$q_{ij,\underline{t}\,\underline{\tau}} = (\tau_{j+k} - \tau_j) \sum_{\ell=i}^{j} \varphi_{i,k,\underline{t}}(t_{\ell+k})/\varphi'_{j-1,k+2,\underline{\tau}}(\tau_{\ell+k})$$

$$= \sum_{\ell=i}^{j} (\tau_{j+k} + \tau_\ell - \tau_\ell + \tau_j)\varphi_{i,k,\underline{t}}(t_{\ell+k})/\varphi'_{j-1,k+2,\underline{\tau}}(\tau_{\ell+k})$$

$$= -\sum_{\ell=i}^{j-1} \varphi_{i,k,\underline{t}}(t_{\ell+k})/\varphi'_{j-1,k+1,\underline{\tau}}(\tau_{\ell+k}) + \sum_{\ell=i}^{j} \varphi_{i,k,\underline{t}}(t_{\ell+k})/\varphi'_{j,k+1,\underline{\tau}}(\tau_{\ell+k}),$$

and consequently

$$\sum_{j=1}^{n} q_{ij,\underline{t}\,\underline{\tau}} = \sum_{j=i}^{k-1} q_{ij,\underline{t}\,\underline{\tau}} = \sum_{\ell=i}^{k-1} \varphi_{i,k,\underline{t}}(t_{\ell+k})/\varphi'_{k-1,k+1,\underline{\tau}}(\tau_{\ell+k}) =: w .$$

Since additionally $\mathrm{int}(\underline{\tau}) = \mathrm{int}(\underline{t})$, then

$$w = \sum_{\ell=i+k}^{2k-1} \frac{\prod_{r=1}^{k-1} (\tau_\ell - t_{r+i})}{\prod_{\substack{r=k \\ r\ne\ell}}^{2k-1} (\tau_\ell - \tau_r)}$$

$$= [\tau_k, \tau_{k+1}, \cdots, \tau_{2k-1}] \prod_{r=1}^{k-1} (\cdot - t_{r+i}) - \sum_{\ell=k}^{i+k-1} \frac{\prod_{r=1}^{k-1} (\tau_\ell - t_{r+i})}{\prod_{\substack{r=k \\ r\ne\ell}}^{2k-1} (\tau_\ell - \tau_r)}$$

$$= 1 - 0 = 1 .$$

The following lemma summarizes this observation.

<u>Lemma 2.2.</u> Let $n \ge 2(k-1)$, and $\mathrm{int}(\underline{\tau}) = \mathrm{int}(\underline{t})$. Then

$$\sum_{j=1}^{n} q_{ij,\underline{t}\,\underline{\tau}} = \sum_{j=1}^{n} q_{ij,\underline{\tau}\,\underline{t}} = 1, \quad \text{all } i .$$

From Lemma 2.1 we conclude

$$A_{\underline{\tau}} = A_{\underline{t}}\, Q_{\underline{t}\,\underline{\tau}} ,$$

and to complete now the proof of the theorem it is enough to show that, for the

particular $\underline{t}$ *and* $\underline{\tau}$ *as specified there,*

$$Q_{\underline{t}\,\underline{\tau}} \geq 0 := (q_{ij,\underline{t}\,\underline{\tau}} \geq 0\ ,\quad \text{all}\quad i,j)$$

since then by Lemma 2.2 and invertible $A_{\underline{t}}$

$$\|A_{\underline{\tau}}^{-1}\|_\infty = \|Q_{\underline{t}\,\underline{\tau}}^{-1}\ A_{\underline{t}}^{-1}\|_\infty \geq \frac{\|A_{\underline{t}}^{-1}\|_\infty}{\|Q_{\underline{t}\,\underline{\tau}}\|_\infty} = \|A_{t}^{-1}\|_\infty \quad .$$

Also (1.9) is confirmed just by changing $A_{\underline{t}}$ and $Q_{\underline{t}\,\underline{\tau}}$ in the previous argument. It was pointed out to me by de Boor [6] that $Q_{\underline{t}\,\underline{\tau}}$ is known to be totally positive if $\underline{t}$ is a refinement of $\underline{\tau}$ on I, i.e. $\text{i.t}(\underline{\tau}) = \text{int}\ (\underline{t})$, $\tau_k \leq t_1$, $\tau_{n+2} \geq t_{n+k}$. The following lemma indicates that this fact holds in the more general situation of the Theorem 1.1, and the columns of $Q_{\underline{t}\,\underline{\tau}}$ deserve to be called discrete B-splines as in [3].

<u>Lemma 2.3.</u> Under assumptions of the Theorem 1.1

$$q_{ij,\underline{t}\,\underline{\tau}} \geq 0\ ,\quad (-)^{j-i}q_{ij,\underline{\tau}\,\underline{t}} \geq 0\ ,\quad \text{all}\quad i,j\ . \tag{2.9}$$

<u>Proof.</u> Choose again $1 \leq i \leq j \leq k-1$. The assumption on $\underline{\tau},\underline{t}$ reads

$$\tau_i \leq t_i \leq t_k < \tau_{i+k}\ ,\qquad 1 \leq i \leq k-1\ . \tag{2.10}$$

From (2.8) we observe that (2.9) holds for $j = i$. Let $j \geq i+1$. The elements $q_{ij,\underline{t}\,\underline{\tau}}$ are linear in any t_ℓ, $i+1 \leq \ell \leq k-1$. Thus from (2.10)

$$\text{sgn}(q_{ij,\underline{t}\,\underline{\tau}}) = \begin{cases} \text{sgn}(q_{ij,\underline{t}\,\underline{\tau}})\big|_{t_\ell = \tau_\ell} \\ \text{or} \\ \text{sgn}(q_{ij,\underline{t}\,\underline{\tau}})\big|_{t_\ell = t_k} \end{cases}$$

or after repeating the argument $k-i-1$ times

$$sgn(q_{ij,\underline{t}\,\underline{\tau}}) = sgn(q_{ij,\underline{t}\,\underline{\tau}})\Big|_{t_{i+1} = \left\{\begin{matrix}\tau_{i+1}\\ \text{or}\\ t_k\end{matrix}\right., \cdots, t_{k-1} = \left\{\begin{matrix}\tau_{k-1}\\ \text{or}\\ t_k\end{matrix}\right.}$$

$$= sgn((\tau_{j+k} - \tau_j)[\tau_{i+k},\tau_{i+k+1},\cdots,\tau_{j+k}]((\cdot - t_k)^m / \prod_{\ell=j}^{m+i} (\cdot - \tau_{\nu_\ell})) \quad (2.11)$$

for some m, $j-i-1 \le m \le k-i-1$, and some $j \le \nu_j < \nu_{j+1} < \cdots < \nu_{i+m} \le k-1$. For $m = j-i-1$ the argument of $sgn(\cdot)$ in (2.11) vanishes, and it is enough to consider $m \ge j-i$. Then

$$[\tau_{i+k}, \tau_{i+k+1},\cdots,\tau_{j+k}]((\cdot - t_k)^m / \prod_{\ell=j}^{m+i} (\cdot - \tau_{\nu_\ell}))$$

$$= [\tau_{i+k}\ \tau_{i+k+1},\cdots,\tau_{j+k}]_z[\tau_{\nu_j},\tau_{\nu_{j+1}},\cdots,\tau_{\nu_{m+i}}]_w \frac{(z-t_k)^m}{z-w}$$

$$= [\tau_{i+k},\tau_{i+k+1},\cdots,\tau_{j+k}]_z \frac{(z - t_k)^m}{(z-\hat{w})^{m+1+i-j}}$$

$$= [\tau_{i+k},\tau_{i+k+1},\cdots,\tau_{j+k}]_z \sum_{r=0}^{m} \binom{m}{r}(z - \hat{w})^{r-m-i+j-1}(\hat{w} - t_k)^{m-r}$$

$$= \frac{1}{(j-i)!} \sum_{r=0}^{m+i-j} \binom{m}{r}(-)^{j-i} \frac{(m-r)!}{(m+i-j-r)!} (\hat{z} - \hat{w})^{r-1-m}(\hat{w} - t_k)^{m-r} + 0$$

$$= (-)^{j-i}\binom{m}{j-i}(\hat{z} - t_k)^{m+i-j}(\hat{w} - t_k)^{j-i}(\hat{z} - \hat{w})^{-m-1}$$

for some $\hat{w},\hat{z}$ with $\tau_j \le \tau_{\nu_j} \le \hat{w} \le \tau_{\nu_{m+i}} \le \tau_{k-1}$, $\tau_{i+k} \le \hat{z} \le \tau_{j+k}$. Thus the first inequality in (2.9) is confirmed. The other follows similarly. ∎

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The B-spline representation of a spline on some interval [a,b] requires the introduction of additional knots outside]a,b[that have nothing in common with the spline space itself. In this note, it is shown that locating all these additional knots at the endpoints a and b of the interval minimizes the matrix norm $\|\cdot\|_\infty$ of the matrix $(\lambda_i N_{j,k})^{-1}$ where λ_i are given linear interpolation conditions, and so is preferable when the B-spline representation of a spline interpolant is to be constructed. Such a choice usually also simplifies the algorithms. In particular, one is able to compute stably the B-spline coefficients of a complete spline

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20. Abstract (continued)

interpolant by Gauss elimination without pivoting though the corresponding matrix fails to be totally positive or diagonally dominant.

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